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A model for the boundary condition of a porous material. Part 1

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In problems where a viscous fluid flows past a porous solid it has frequently been assumed that the tangential component of surface velocity is zero. When the porous solid has an open structure with large pores the external surface stress may produce a tangential flow below the surface. Recently, Beavers & Joseph (1967) have assumed that the surface velocity U_B depends on the mean tangential stress $[\mu(d\bar{u}/dy)]_{y=0}$ in the fluid outside the porous solid through the relation

$$\left[\mu \frac{d\bar{u}}{dy}\right]_{y=0} = \frac{\mu\alpha}{k^{\frac{1}{2}}}(U_B - Q),$$

where Q is the volume flow rate per unit cross-section within the porous material due to the pressure gradient, k is the Darcy constant and α is a constant which depends only on the nature of the porosity. An artificial mathematical model of a porous medium is proposed for which the flow can be calculated both inside and outside the surface. This conceptual model was materialized and the experimental results agree with the calculations. The calculated values of α so found are not quite independent of the external means of producing the external tangential stress.

1. Introduction

Experiments by Beavers & Joseph (1967) have shown that, when a viscous fluid flows past a porous solid, tangential stress moves the fluid close below the surface with velocity U_B which is slightly greater than Q, the velocity of the fluid in the bulk of the porous medium. They measured this difference by confining the fluid above the porous surface to a narrow channel of height h, and they found a slightly greater flow than would have occurred if U_B were equal to Q. To analyze their measurements they assumed that $U_B - Q$ is related to the surface drag $\mu(du/dy)_{y=0}$ by the equation

$$\left(\frac{du}{dy}\right)_{y=0} = \frac{\alpha}{k^{\frac{1}{2}}}(U_B - Q), \tag{1}$$

where k is the Darcy constant, namely the ratio $\mu Q/(dp/dx)$, where Q is the volumetric flow rate through unit area perpendicular to x under the action of a pressure gradient dp/dx. α is a constant of the material which is assumed not to depend on h. It is reasonable to make this assumption as a limiting condition when h

is large compared, say, with $k^{\frac{1}{2}}$, but some justification is required for the assumption that α depends only on the material and not on other features of the geometry of the measuring apparatus. For this reason it seemed desirable to invent an ideal porous material for which both k and α could be calculated and compare the results of calculations with experiment. Such a material, consisting of parallel plates, has already been described (Taylor 1960) and the amount of fluid left behind when it is saturated with fluid and moved over a plane surface has been

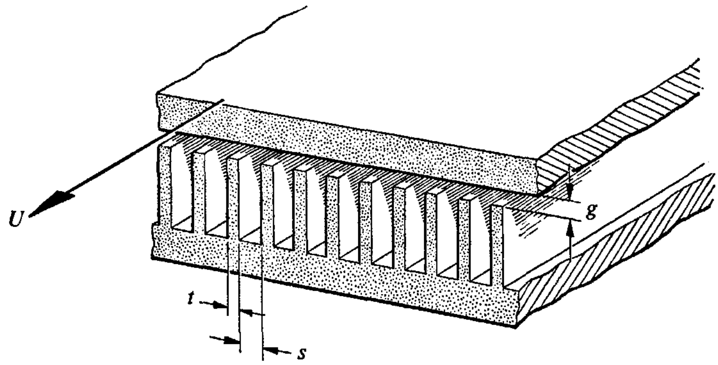

FIGURE 1

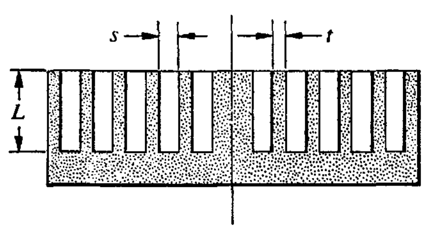

FIGURE 2

calculated. In this case the ideal material was in contact with the plane. When there is a gap, g, between the porous surface and the plane, the analysis can still be carried out, but it is much more complicated than in the case of the 'mathematical paint brush' just described. The analysis is similar to that given by Cockcroft (1927) in the context of an electrostatic problem but the parameters required for the present purpose are not the same as those discussed by Cockcroft. The necessary calculations have been carried out by Dr S. Richardson and are described by him in part 2 of this paper.

The model to which the analysis applies is shown in figure 1 which also indicates the symbols used. A flat plate moves with velocity U parallel to the sheets of which the model is composed. The symbol u is used for the fluid velocity to conform to Beavers & Joseph's notation, but the analysis of part 2 is based on taking the plane perpendicular to this as (x, y). The experiments to be described later were made with a disk (figure 2), containing concentric grooves of depth L

which was only 4 times their width s but the calculations show that there would be no significant difference if the depth of the grooves were infinite, as is assumed for the porous model.

Beavers & Joseph point out that their method could not be expected to yield meaningful results when the grain sizes of the material are comparable with the height h of their channel. When there is no mean pressure gradient, as in the present experiment, the mean shear stress over planes parallel to the porous surface must be independent of y. If σ_{yz} is the shear stress at any point and inertia effects are negligible

$$\sigma_{yz} = \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial z}\right), \tag{2}$$

where v is the component of velocity parallel to y and u that parallel to z. The mean shear stress over planes perpendicular to the y axis is

$$\overline{\sigma_{yz}} = \mu\left(\frac{\partial \overline{u}}{\partial y}+\frac{\partial \overline{v}}{\partial z}\right) = \mu\frac{d\overline{u}}{dy} \tag{3}$$

and this is true over any plane which does not actually intersect any possible protuberances from the porous material. For this reason $d\overline{u}/dy$ is independent of y even when the grains are comparable with g.

The mean surface velocity U_B has a meaning even when protuberances from the porous material are large and it can be determined by measuring the tangential stress on a smooth plate as it moves at height g above the porous surface. For this reason values of α, as defined in (1), are meaningful even though the gap g, which is analogous to Beavers & Joseph's h, is smaller than the distance s between the planes of the porous model.

2. Experimental apparatus for measuring shear stress

As already indicated, it was not practical to measure the tangential reaction between the flat plate and the porous model (figure 1). The grooved Perspex disk sketched in figure 2 was used. An accurately made circular Perspex dish A (figure 3) of 14·2 cm internal radius was mounted centrally on a horizontal table which could rotate about a vertical axis, and was driven by a constant speed motor M through a variable reduction gear V. The grooved Perspex disk C (figures 2 and 3) of radius R = 5·5 cm was mounted centrally on the bottom of the dish A. This disk contained 17 grooves 0·8 cm deep and 0·2 cm wide separated by cylindrical tongues 0·1 cm thick and it was set with its upper surface as accurately horizontal as was possible, with a spirit level. Above it was suspended a brass disk D by means of a steel torsion wire E (30·7 cm long, 0·023 cm diameter). Though this disk and its spindle H were turned in one piece and the wire was central, the disk had to be balanced by adding small weights before it could swing freely on its wire with only 0·004 cm clearance at all points over the grooved disk C. The upper end of the wire E was fixed in a circular protractor F. This could rotate on a horizontal table T which could be raised or lowered by means of a screw, S, with a 1/16 in. pitch and a graduated head J. To ensure that the height of the brass disk above the grooved surface was known, a micrometer was fixed to a ground brass bar which could slide over the edge of the dish A. A conical point was fitted to the micrometer which could be screwed down till the point just met its image in the top of the brass disk D. That this measurement could be made without exerting any appreciable force on the disk was ensured by finding that the rise in the top of the torsion wire when it was turned was the same as that measured by the micrometer.

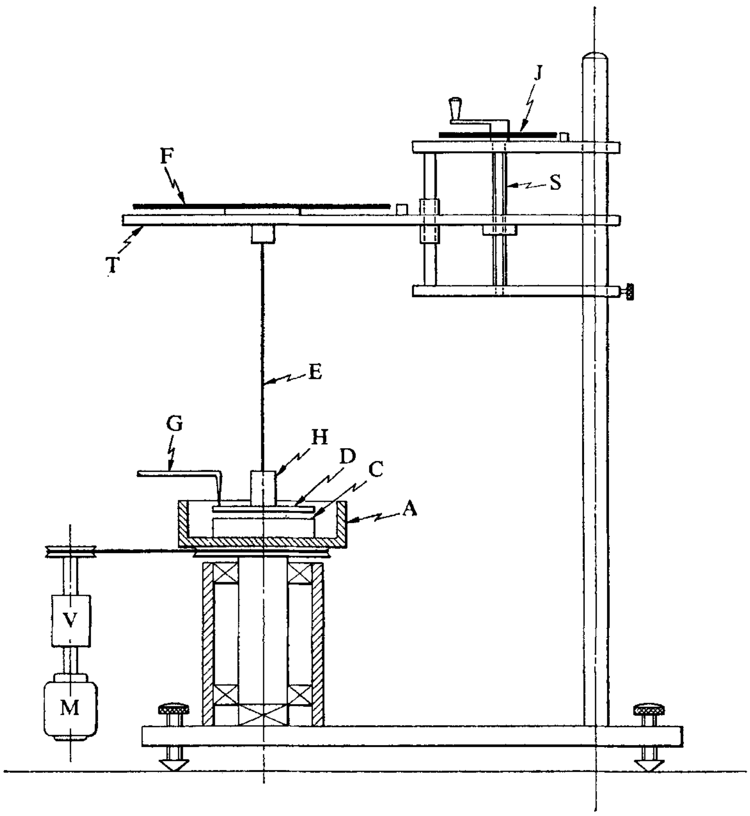

FIGURE 3

The angle of twist of the torsion wire was measured by reading the graduated head F and the brass disk was always brought to a fixed position by turning F till a radial line scribed on the top of the disk was under a fixed pointer G.

3. Method of performing an experiment

The dish A was filled with a Shell oil described as Talpa to a level 1 mm above the grooved disk. The brass disk hanging on its torsion wire was lowered onto the oil and, making sure that no bubbles had been trapped, it was allowed to settle for a day with the torsion wire loose so as to ensure that it was resting on the grooved disk. The wire was then very slowly tightened till the micrometer showed that the disk had begun to rise. As soon as this happened the brass disk would very slowly turn to a fixed position and the pointer G was set to mark it. That the lower face of the brass disk was horizontal was ensured because it very slowly returned to a fixed position after turning the dish A through an angle and leaving it stationary. This condition occurred as soon as the micrometer showed that the brass disk had lifted through 0·004 cm. The first measurements were made when the gap was 1/160th inch = 0·0159 cm. The motor was started so that the grooved disk rotated once in $T = 123$ seconds. The torsion head F was rotated through $\phi = 40{\cdot}0°$ till it brought the radial mark back under the pointer G.

g (cm)	0·0159	0·0318	0·0477	0·0636	0·0777
$T\phi°$	4887	3338	2351	1951	1601
$\dfrac{(t+s)\,G}{\pi\mu\Omega R^4}$	4·40	3·00	2·115	1·723	1·440
$\dfrac{t+s}{g}$	18·9	9·43	6·3	4·72	3·86

TABLE 1

The dish A (figure 3) was made to rotate at a number of different speeds so that the time of revolution T covered the range between 123 and 31·5 sec and it was found that in that range $T \times \phi°$ was nearly constant with mean value 4887. The measurement was repeated with $g = 2$, 3, 4 and 5 times 1/160 in. and the resulting value of $T\phi°$ given in line 2 of table 1. For the figures in line 3, G was taken as $196{\cdot}9\phi°$ and μ as 3·64 poise.

4. Comparison with the calculation in part 2

The torque, G, exerted on the brass disk by the rotating grooved disk has been calculated on the assumption that the tangential stress at radius r is the same as that which would exist in the linear model (figure 1) when a flat sheet distant g from the grooved sheet moves with velocity Ωr, Ω being the angular velocity. The calculations in part 2 give $(t+s)\,G/\pi\mu\Omega R^4$ as a function of $(t+s)/g$ in equations (2.14), (2.15) and (2.16). For the particular case $s/t = 2$, for which the experiments were made, the results are shown in figure 4. If the disk had no grooves the simple calculation for a plane disk rotating at a distance g above a flat plate shows that $G/\pi\mu\Omega R^4 = 1/2g$ and this relationship is shown in figure 4 by the straight line. The difference between the straight line and the curve is due to the fact that U_B is not zero as it would be if the disk had been flat and impermeable.

To measure the torque G the torsional stiffness of the wire E was determined by a well-known method. The period of oscillation of the brass disk was first measured. A body of known moment of inertia was then placed on it and the period of oscillation again measured. The result was that when the wire twisted through an angle ϕ degrees the torque exerted by it on the brass disk was $G = 196{\cdot}9\phi$ gm cm^2 sec^{-2}. Except for the first measurement recorded in line 2 of table 1 all were made at temperatures between 23·3 and 23·5 °C. The viscosity of the Shell Talpa oil used was determined by the capillary tube method at 23·4° as 3·64 poise. The angular velocity was measured by timing the passage of a mark on the turntable. If T sec is this period $\Omega = 2\pi/T$. The calculated values of

$(t+s)\,G/\mu\pi\Omega R^4$ are shown as a function of $(t+s)/g$ in figure 4 and the measured values by crosses. The agreement is good except in the case $g = 0{\cdot}0159$ cm. When this measurement was made it was not appreciated that Talpa oil changes its viscosity nearly 10% per degree at 20° C and sufficiently accurate temperature measurement may not have been made.

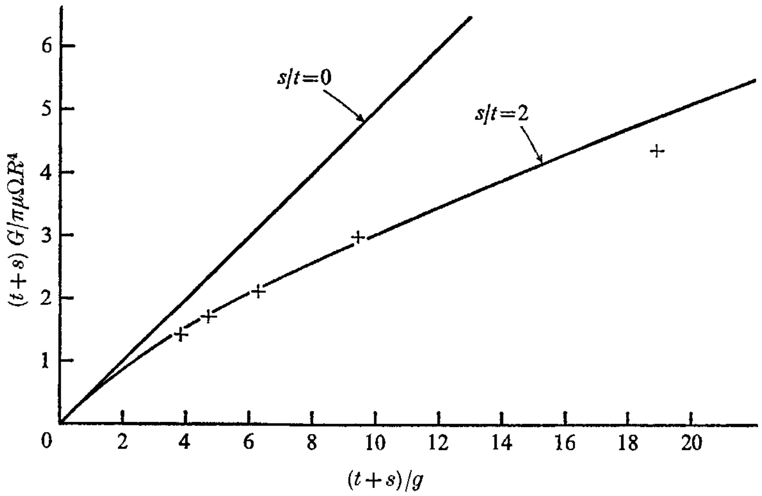

Figure 4

5. Calculation of α

The primary object of this exercise was to find how far the assumption that α, as defined in (1), depends only on the structure of the porous material, is true. In the ideal porous model of figure 1 a calculation of flow between plates separated by a distance s shows that the Darcy constant is $k = s^3/12\,(s+t)$, which in the experiment was $2{\cdot}22 \times 10^{-3}\,\text{cm}^2$ so that $k^{\frac{1}{2}} = 4{\cdot}7 \times 10^{-2}$ cm. Since there was no pressure gradient in my experiment $Q = 0$ and for the reason already given

$$d\bar{u}/dy = (U - U_B)/g$$

so that

$$\alpha = \frac{k^{\frac{1}{2}}}{g}\left(\frac{U-U_B}{U_B}\right). \tag{4}$$

Calculated values of $(U - U_B)/U$ for the model are shown in figure 5 as a function of $g/(s+t)$ for the case $s/t = 2$. Writing $(U - U_B)/U = P$, (4) becomes

$$\frac{\alpha}{k^{\frac{1}{2}}} = \frac{P}{g(1-P)} \tag{5}$$

and the calculated values of α are shown as a function of $g/(s+t)$ in figure 6 for the case $s/t = 2$. It will be seen that as g increases there is a rapid decrease in α and when g has reached $0{\cdot}5(s+t) = 1{\cdot}5$ mm in the experiment α has nearly reached its asymptotic value 2·035. Since, by definition, α is associated with $d\bar{u}/dy$, and $d\bar{u}/dy$ is constant through the whole region between the upper plate and the plane which touches the highest point in the surface of the porous medium, the definition of α is meaningful even when the porous surface is rough. Figure 6 shows that in experiments to determine α the height g of the plane which gives rise to the velocity gradient need only be about half the spacing of neighbouring plates above

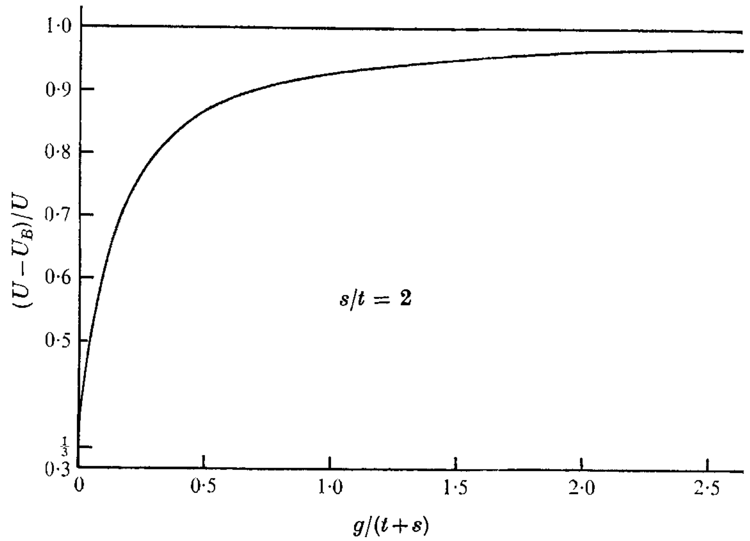

FIGURE 5

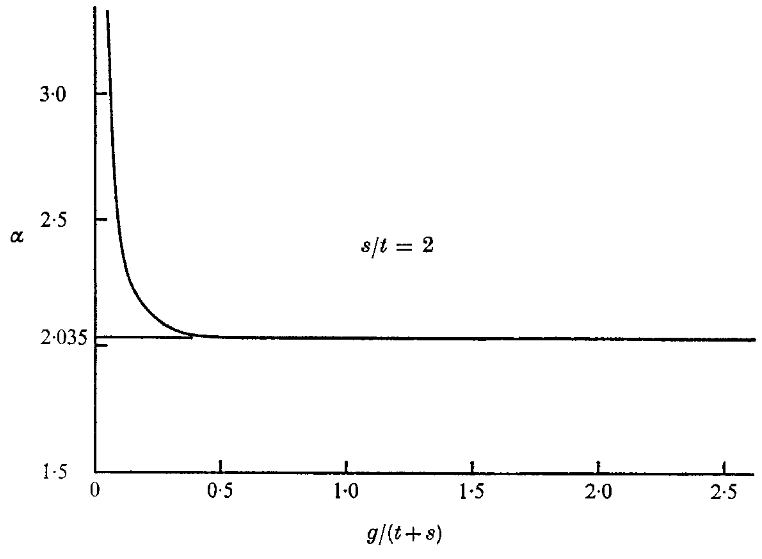

FIGURE 6

the model to ensure an accurate measurement of the limiting value and that measurements made with greater spacing give values which depend only on the properties of the porous material. The theoretical limiting values of α for different values of the porosity of the model are shown in figure 7. The limiting value of α as $t \rightarrow 0$ and the porosity tends to 1·0 is 1·308.

It is worth noticing that if the resistance of a very porous material could be regarded as due to a very sparse distribution of points at each of which the fluid exerts a force $A\mu u$, the corresponding value of α is 1·0. The equation for u in this case is

$$-\frac{dp}{dx}+\mu\frac{d^2u}{dy^2} = nA\mu u, \tag{6}$$

where n is the number of resistance points per unit volume. Darcy's law is expressed by the equation

$$nA = k^{-1} \tag{7}$$

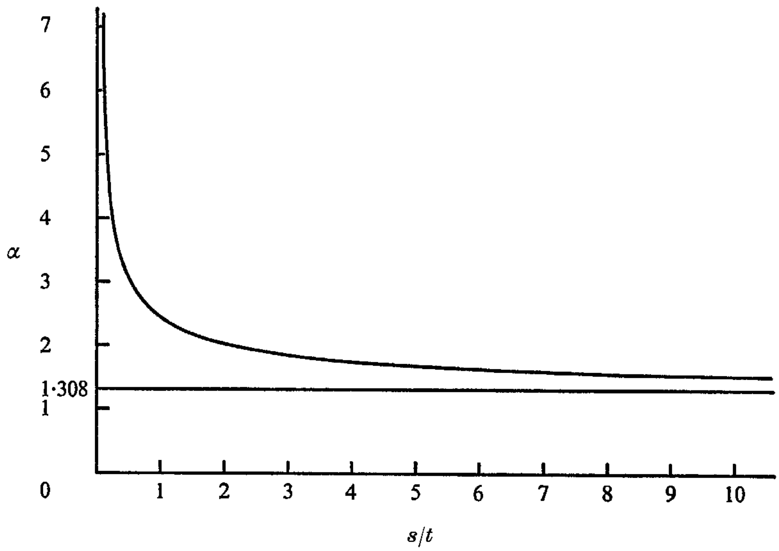

FIGURE 7. Limiting value of α as $t \to 0$.

and the velocity Q of the flow inside the material far from the surface is

$$Q = -\frac{k}{\mu}\frac{dp}{dx}. \tag{8}$$

The solution of (6) for which $u = Q$ when $y \to -\infty$ is

$$u - Q = (U_B - Q)\, e^{y/k^{\frac{1}{2}}},$$

so that

$$\left[\frac{du}{dy}\right]_{y=0} = \frac{1}{k^{\frac{1}{2}}}(U_B - Q) \tag{9}$$

and comparing (9) with (1) it will be seen that for this model $\alpha = 1{\cdot}0$.

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